ON THE CHERN NUMBER OF AN IDEAL

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ABSTRACT. We settle the negativity conjecture of Vasconcelos for the Chern number of an ideal in certain unmixed quotients of regular local rings by explicit calculation of the Hilbert polynomials of all ideals generated by systems of parameters.

INTRODUCTION

Let I be an $\mathfrak{m}$-primary ideal in a local ring $(R, \mathfrak{m})$ of dimension d. Let $H(I, n) = \lambda(R/I^n)$ denote the Hilbert function of I where $\lambda(M)$ denotes the length of an R-module M. The Hilbert function $H(I, n)$ is given by a polynomial $P(I, n)$ of degree d for large n. It is written in the form

$$P(I, n) = e_0(I)\binom{n + d - 1}{d} - e_1(I)\binom{n + d - 2}{d - 1} + \cdots + (-1)^d e_d(I).$$

If I is generated by a system of parameters, then R is Cohen-Macaulay if and only if $e_0(I) = \lambda(R/I)$. Recently it has been observed by Vasconcelos [7] that the signature of the coefficient $e_1(I)$, called the *Chern number* of I, can be used to characterize Cohen-Macaulay property of R for large classes of rings. In the Yokohama Conference in 2008 Vasconcelos proposed the following:

Negativity Conjecture: Let $(R, \mathfrak{m})$ be an unmixed, equidimensional local ring which is a homomorphic image of a Cohen-Macaulay local ring. Then R is not Cohen-Macaulay if and only if for any ideal J generated by a system of parameters, $e_1(J) < 0$.

Ghezzi, Hong and Vasconcelos [3] settled the conjecture for (1) Noetherian local domains of dimension $d \geq 2$, which are homomorphic images of Cohen-Macaulay local rings and for (2) Universally catenary integral domains containing a field. The Negativity Conjecture has been resolved for all unmixed local rings by S. Goto recently.

In this paper we settle the Negativity Conjecture for certain unmixed quotients of regular local rings, by explicitly finding the Hilbert polynomial of all parameter ideals.

Acknowledgements: We wish to thank Shiro Goto, A. Simis and V. Srinivas for several useful conversations.

1. The Hilbert polynomial of parameter ideals in certain quotients of regular local rings

L. Ghezzi, J. Hong and W. Vasconcelos [3] calculated the Chern number of any parameter ideal in certain quotients of regular local rings of dimension four. We recall their result first:

Theorem 1. *Let $(S, \mathfrak{m})$ be a four dimensional regular local ring with $S/\mathfrak{m}$ infinite. Let $P_1, P_2, \ldots, P_r$ be a family of height two prime ideals of S so that for $i \neq j, P_i + P_j$ is $\mathfrak{m}$-primary. Put $R = S/\cap_{i=1}^r P_i$. Let J be an $\mathfrak{m}$-primary parameter ideal of R. Let $L = \oplus_{i=1}^r S/P_i/R$. If $J \subseteq \operatorname{ann} L$ then $e_1(J) = -\lambda(L)$ and $e_2(J) = 0$.*

In this section we find the Hilbert polynomials of parameter ideals in R when S is any regular local ring.

Lemma 2. *Let $(S, \mathfrak{n})$ be an r-dimensional regular local ring. Let I be a height h Cohen-Macaulay ideal of S. Suppose $a_1, \ldots, a_d \in S$ such that $(a_1 + I, \ldots, a_d + I)$ is a system of parameters in S/I. Then $a_1, \ldots, a_d$ is a regular sequence in S.*

Proof. Let $J = (a_1, \ldots, a_d)$. Then $\lambda(S/I \otimes_S S/J) < \infty$. Hence by Serre's theorem [6, Theorem 3, Chapter 5]

$$\dim S/I + \dim S/J \leq \dim S = r.$$

As S is regular, it is catenary. Thus $r - \operatorname{ht} I + r - \operatorname{ht} J \leq r$. Therefore $d \leq \operatorname{ht} J \leq d$. Hence $\operatorname{ht} J = d$ and consequently $a_1, \ldots, a_d$ is an S-regular sequence. $\square$

Lemma 3. *Let J, I and S be as above. Then for all $j, n \geq 1$*

$$\operatorname{Tor}_j^S(S/J^n, S/I) = 0.$$

Proof. We will apply induction on n. Let $n = 1$. As J is a complete intersection, the Koszul complex $K(\underline{a})$ of the sequence $\underline{a} = a_1, a_2, \ldots, a_d$

$$K(\underline{a}) : 0 \longrightarrow S \longrightarrow S^d \longrightarrow S^{\binom{d}{2}} \longrightarrow \ldots \longrightarrow S^d \longrightarrow S \longrightarrow S/J \longrightarrow 0$$

gives a free resolution of S/J. Tensoring the above complex with $R := S/I$ we get

$$K(\underline{a}, S/I) : 0 \longrightarrow R \longrightarrow R^d \longrightarrow R^{\binom{d}{2}} \longrightarrow \ldots \longrightarrow R^d \longrightarrow R \longrightarrow R/K \longrightarrow 0$$

which is the Koszul complex of $JR = K$. As K is generated by an R-regular sequence, the above is a free resolution of R/K. Hence $\operatorname{Tor}_j^S(S/J, S/I) = 0$ for all $j \geq 1$. Since J is generated by a regular sequence, J^n/J^{n+1} is a free S/J-module. Consider the exact sequence

$$0 \longrightarrow J^n/J^{n+1} \longrightarrow S/J^{n+1} \longrightarrow S/J^n \longrightarrow 0.$$

This gives rise to the long exact sequence

$$\ldots \longrightarrow \operatorname{Tor}_j^S(J^n/J^{n+1}, S/I) \longrightarrow \operatorname{Tor}_j^S(S/J^{n+1}, S/I) \longrightarrow \operatorname{Tor}_j^S(S/J^n, S/I) \longrightarrow \ldots$$

By induction on n, it follows that $\operatorname{Tor}_j^S(S/J^{n+1}, S/I) = 0$ for all $j \geq 1$. $\qquad\square$

Lemma 4. *Let $J = (a_1, \ldots, a_d)$ be a complete intersection of height d in a regular local ring $(R, \mathfrak{m})$. Let L be an R-module of finite length. Then $\mathcal{R}(J) \otimes_R L$ is a finite $\mathcal{R}(J)$-module of dimension d and*

$$\operatorname{Supp}(\mathcal{R}(J) \otimes_R L) = V(\mathfrak{m}\mathcal{R}(J)).$$

Proof. Suppose $\mathcal{P} \in \operatorname{Supp}(\mathcal{R}(J) \otimes_R L)$ and $p = \mathcal{P} \cap R$. Then $(\mathcal{R}(J) \otimes_R L)_{\mathcal{P}} = (\mathcal{R}(J_p) \otimes_{R_p} L_p)_{\mathcal{P}} \neq 0$. Hence $L_p \neq 0$. As L has finite length, $p = \mathfrak{m}$. Hence $\mathfrak{m}\mathcal{R}(J) \subseteq \mathcal{P}$. Since $\mathcal{R}(J)/\mathfrak{m}\mathcal{R}(J) \simeq R/\mathfrak{m}[T_1, T_2, \ldots, T_d]$ is a polynomial ring, $\mathfrak{m}\mathcal{R}(J)$ is a prime ideal of $\mathcal{R}(J)$.

We will prove the other inclusion by induction on $\lambda(L)$. Let $\lambda(L) = 1$. Then $L \cong R/\mathfrak{m}$. Therefore $\mathcal{R}(J) \otimes_R L = \mathcal{R}(J) \otimes_R R/\mathfrak{m} = \mathcal{R}(J)/\mathfrak{m}\mathcal{R}(J)$. Hence $\mathfrak{m}\mathcal{R}(J) \in \operatorname{Supp}(\mathcal{R}(J) \otimes_R L)$. Now assume that $\lambda(L) > 1$. Then we have the following exact sequence of R-modules

$$0 \longrightarrow R/\mathfrak{m} \xrightarrow{i} L \longrightarrow C \longrightarrow 0 \tag{1}$$

where $C = \operatorname{coker} i$. Tensoring (1) with $\mathcal{R}(J)$ we get the exact sequence

$$\mathcal{R}(J) \otimes_R R/\mathfrak{m} \longrightarrow \mathcal{R}(J) \otimes_R L \longrightarrow \mathcal{R}(J) \otimes_R C \longrightarrow 0. \tag{2}$$

Localize the above sequence at $\mathfrak{m}\mathcal{R}(J)$ to get $\mathfrak{m}\mathcal{R}(J) \in \operatorname{Supp}(\mathcal{R}(J) \otimes_R L)$ using induction. $\qquad\square$

Lemma 5. *Let S be a local ring, $a_1, \ldots, a_d$ be a regular sequence and $J = (a_1, \ldots, a_d)$. Let L be an S-module of finite length. If $J \subseteq \operatorname{ann} L$ then*

$$\lambda(\operatorname{Tor}_1(L, S/J^n)) = \binom{n + d - 1}{d - 1} \lambda(L).$$

Proof. By [5, Example 10] for any $n > 0$, J^n is generated by the maximal minors of the $n \times (n + d - 1)$ matrix A where

$$A = \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_d & 0 & 0 & \cdots & 0 \\ 0 & a_1 & a_2 & \cdots & a_{d-1} & a_d & 0 & \cdots & 0 \\ 0 & 0 & a_1 & \cdots & a_{d-2} & a_{d-1} & a_d & \cdots & 0 \\ 0 & 0 & 0 & \cdots & a_1 & a_2 & a_3 & \cdots & a_d \end{pmatrix}$$

By Eagon-Northcott [2, Theorem 2], the minimal free resolution of S/J^n is given by

$$0 \longrightarrow S^{\beta_d} \longrightarrow S^{\beta_{d-1}} \longrightarrow \cdots \longrightarrow S^{\beta_1} \longrightarrow S \longrightarrow S/J^n \longrightarrow 0 \qquad (3)$$

where the Betti numbers of S/J^n are given by

$$\beta_i^S(S/J^n) = \binom{n+d-1}{d-i}\binom{n+i-2}{i-1}, \ 1 \le i \le k.$$

Taking tensor product of (3) with L we get the following complex

$$0 \longrightarrow L^{\beta_d} \longrightarrow L^{\beta_{d-1}} \longrightarrow \cdots \longrightarrow L^{\beta_1} \longrightarrow L \longrightarrow L/J^n L \longrightarrow 0.$$

Since $J \subseteq \operatorname{ann} L$, the maps in the above complex are zero. Hence

$$\lambda(\operatorname{Tor}_1(L, S/J^n)) = \beta_1 \lambda(L) = \binom{n+d-1}{d-1}\lambda(L).$$

$\square$

Theorem 6. *Let $(S, \mathfrak{n})$ be a regular local ring of dimension r and $I_1, \ldots, I_g$ be Cohen-Macaulay ideals of height h which satisfy the condition: $I_i + I_j$ is $\mathfrak{n}$-primary for $i \ne j$. Let $R = S/I_1 \cap \ldots \cap I_g$ and $d = \dim R \ge 2$. Let $a_1, \ldots, a_d \in S$ such that their images in R form a system of parameters. Let $J = (a_1, \ldots, a_d)$, $L = [\oplus_{i=1}^g S/I_i]/R$ and $K = JR$. Put $H_J(L, n) = \lambda(J^n \otimes_R L)$ and $P_J(L, n)$ be the corresponding Hilbert polynomial. Then*

(1) $P_J(L, n) = -e_1(K)\binom{n+d-2}{d-1} + e_2(K)\binom{n+d-3}{d-2} - \cdots + (-1)^d e_d(K) + \lambda(L).$

(2) *If $J \subseteq \operatorname{ann} L$ then*

$$P(K, n) = e_0(K)\binom{n+d-1}{d} + \lambda(L)\binom{n+d-2}{d-1} + \lambda(L)\binom{n+d-3}{d-2} + \cdots + n\lambda(L).$$

Proof. First we show that $\lambda(L) < \infty$. Consider the exact sequence

$$0 \longrightarrow \frac{S}{I_1 \cap \cdots \cap I_g} \longrightarrow N = \frac{S}{I_1} \oplus \ldots \oplus \frac{S}{I_g} \longrightarrow L \longrightarrow 0. \qquad (4)$$

Let P be a non-maximal prime ideal of S not containing any $I_1, \ldots, I_g$. Then $L_P = 0$. If there is an i such that $I_i \subseteq P$, then for $j \ne i$, $I_j \not\subseteq P$. Hence

$$(S/I_1 \cap \ldots \cap I_g)_P = (S/I_1 \oplus \ldots \oplus S/I_g)_P = (S/I_i)_P.$$

Thus $L_P = 0$. Hence Supp $L = \{\mathfrak{n}\}$. Thus $\lambda(L) < \infty$.

By the depth lemma, depth $R = 1$. Thus R is not Cohen-Macaulay. Tensoring (4) with S/J^n we get the exact sequence

$$\longrightarrow \operatorname{Tor}_1^S(R, S/J^n) \longrightarrow \bigoplus_{i=1}^{g} \operatorname{Tor}_1^S(S/I_i, S/J^n) \longrightarrow \operatorname{Tor}_1^S(L, S/J^n)$$

$$\longrightarrow R/K^n \longrightarrow \bigoplus_{i=1}^{g} S/(I_i, J^n) \longrightarrow L/J^n L \longrightarrow 0.$$

By Lemma 3, $\operatorname{Tor}_1^S(S/I_i, S/J^n) = 0$ for all i, n. For large n, $J^n L = 0$ as $\lambda(L) < \infty$. For large n,

$$\lambda(R/K^n) = e_0(K)\binom{n+d-1}{d} - e_1(K)\binom{n+d-2}{d-1} + \cdots + (-1)^d e_d(K).$$

By (4) and additivity of $e_0(J, \_)$ we get $e_0(K) = \sum_{i=1}^{g} e_0(J, S/I_i)$. Hence

$$\lambda(\operatorname{Tor}_1^S(L, S/J^n)) - \lambda(L) = \sum_{i=1}^{d}(-1)^i e_i(K)\binom{n+d-1-i}{d-i}.$$

From the exact sequence

$$0 \longrightarrow J^n \longrightarrow S \longrightarrow S/J^n \longrightarrow 0$$

we get

$$\longrightarrow \operatorname{Tor}_1^S(J^n, L) \longrightarrow \operatorname{Tor}_1^S(S, L) \longrightarrow \operatorname{Tor}_1^S(S/J^n, L)$$

$$\longrightarrow J^n \otimes_S L \longrightarrow S \otimes L \longrightarrow L/J^n L \longrightarrow 0.$$

Hence for large n,

$$\lambda(\operatorname{Tor}_1^S(S/J^n, L)) = \lambda(J^n \otimes_S L) = \left[\sum_{i=1}^{d}(-1)^i e_i(K)\binom{n+d-1-i}{d-i}\right] + \lambda(L). \quad (5)$$

Since $\dim(\mathcal{R}(J) \otimes_R L) = d$ and $d \geq 2$, $e_1(K) < 0$. If $J \subseteq \operatorname{ann} L$ then by Lemma 5

$$\lambda(\operatorname{Tor}_1(L, S/J^n)) = \binom{n+d-1}{d-1}\lambda(L).$$

Substituting in (5) we get

$$\binom{n+d-1}{d-1}\lambda(L) - \lambda(L) = -e_1(K)\binom{n+d-2}{d-1} + e_2(K)\binom{n+d-3}{d-2} - \cdots + (-1)^d e_d(K).$$

Using the equation,

$$\binom{n+d-1}{d-1} = 1 + \sum_{i=1}^{d-1}\binom{n+d-i-1}{d-i}$$

we obtain $e_i(K) = (-1)^i \lambda(L)$ for $i = 1, 2, \ldots, d-1$ and $e_d(K) = 0$.

$\square$

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